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Linear choosability of sparse graphs

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ABSTRACT

A linear coloring is a proper coloring such that each pair of color classes induces a union of disjoint paths. We study the linear list chromatic number, denoted $\mathrm{lc}_\ell(G)$, of sparse graphs. The maximum average degree of a graph G, denoted $mad(G)$, is the maximum of the average degrees of all subgraphs of G. It is clear that any graph G with maximum degree $\Delta(G)$ satisfies $\mathrm{lc}_\ell(G) \geq \lceil \Delta(G)/2 \rceil + 1$. In this paper, we prove the following results: (1) if $\mathrm{mad}(G) < 12/5$ and $\Delta(G) \geq 3$, then $\mathrm{lc}_\ell(G) = \lceil \Delta(G)/2 \rceil + 1$, and we give an infinite family of examples to show that this result is best possible; (2) if $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 9$, then $\mathrm{lc}_\ell(G) \leq \lceil \Delta(G)/2 \rceil + 2$, and we give an infinite family of examples to show that the bound on $\mathrm{mad}(G)$ cannot be increased in general; (3) if G is planar and has girth at least 5, then $\mathrm{lc}_\ell(G) \leq \lceil \Delta(G)/2 \rceil + 4$.

1. Introduction

In 1973, Grünbaum introduced *acyclic colorings* [3], which are proper colorings with the additional property that each pair of color classes induces a forest. In 1997, Hind, Molloy, and Reed introduced frugal colorings [4]. A proper coloring is *k-frugal* if the subgraph induced by each pair of color classes has maximum degree less than k. Yuster [8] combined the ideas of acyclic coloring and 3-frugal coloring in the notion of a *linear coloring*, which is a proper coloring such that each pair of color classes induces a union of disjoint paths—also called a *linear forest*. We write $\mathrm{lc}(G)$ to denote the *linear chromatic number* of G, which is the smallest integer k such that G has a proper k-coloring in which every pair of color classes induces a linear forest.

We begin by noting easy upper and lower bounds on $\mathrm{lc}(G)$. If G is a graph with maximum degree $\Delta(G)$, then we have the naive lower bound $\mathrm{lc}(G) \geq \lceil \Delta(G)/2 \rceil + 1$, since each color can appear on at most two neighbors of a vertex of maximum degree. Observe that $\mathrm{lc}(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1$, where $\chi(G)$ denotes the chromatic number of G and G^2 is the square graph of G. Yuster [8] constructed an infinite family of graphs such that $\mathrm{lc}(G) \geq C_1 \Delta(G)^{3/2}$, for some constant C_1. He also proved an upper bound of $\mathrm{lc}(G) \leq C_2 \Delta(G)^{3/2}$, for some constant C_2 and for sufficiently large $\Delta(G)$.

Note that trees with maximum degree $\Delta(G)$ have linear chromatic number $\lceil \Delta(G)/2 \rceil + 1$, i.e., the naive lower bound holds with equality (for example, we can color greedily in the order of a breadth-first search from an arbitrary vertex). This equality for trees suggests that sparse graphs might have linear chromatic number close to the naive lower bound. To be more precise: Does there exist a constant C such that every sparse graph G satisfies $\mathrm{lc}(G) \leq \lceil \Delta(G)/2 \rceil + C$? To state the previous results related to this question, we first introduce some more notation.

We start with linear list colorings, which are linear colorings from assigned lists. Formally, let $\mathrm{lc}_\ell(G)$ be the *linear list chromatic number* of G, that is, the smallest integer k such that if each vertex $v \in V(G)$ is given a list $L(v)$ with $|L(v)| \geq k$,

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then G has a linear coloring such that each vertex v gets a color $c(v)$ from its list $L(v)$. When all the lists are the same, linear list coloring is the same as linear coloring. General list coloring was first introduced by Erdös, Rubin, and Taylor [1] and independently by Vizing [7] in the 1970s, and it has been well explored since then [5].

Linear list colorings were first studied by Esperet, Montassier, and Raspaud [2]. The *maximum average degree* of a graph G, denoted $\text{mad}(G)$, is the maximum of the average degrees of all of its subgraphs, i.e., $\text{mad}(G) = \max_{H\subseteq G} \frac{2|E(H)|}{|V(H)|}$. Observe that the family of all trees is precisely the set of connected graphs with $\text{mad}(G) < 2$ (so indeed we are generalizing our motivating example, trees). The following results were shown in [2]:

Theorem A *([2]). Let G be a graph:*

(1) *If* $\text{mad}(G) < 8/3$, *then* $\text{lc}_\ell(G) \le \lceil \Delta(G)/2 \rceil + 3$.
(2) *If* $\text{mad}(G) < 5/2$, *then* $\text{lc}_\ell(G) \le \lceil \Delta(G)/2 \rceil + 2$.
(3) *If* $\text{mad}(G) < 16/7$ *and* $\Delta(G) \ge 3$, *then* $\text{lc}_\ell(G) = \lceil \Delta(G)/2 \rceil + 1$.

The *girth* of a graph G, denoted $g(G)$, or simply g, is the length of its shortest cycle. By an easy application of Euler's formula, we see that every planar graph G with girth g satisfies $\text{mad}(G) < 2g/(g-2)$. So we can obtain some results on planar graphs from the above results. Raspaud and Wang [6] proved somewhat stronger results for planar graphs.

Theorem B *([6]). Let G be a planar graph:*

(1) *If* $g(G) \ge 5$, *then* $\text{lc}(G) \le \lceil \Delta(G)/2 \rceil + 14$.
(2) *If* $g(G) \ge 6$, *then* $\text{lc}(G) \le \lceil \Delta(G)/2 \rceil + 4$.
(3) *If* $g(G) \ge 13$ *and* $\Delta(G) \ge 3$, *then* $\text{lc}(G) = \lceil \Delta(G)/2 \rceil + 1$.

Our goal in the paper is to improve the results in the above two theorems. We prove the following.

Theorem 1. *Let G be a graph:*

(1) *If G is planar and has* $g(G) \ge 5$, *then* $\text{lc}_\ell(G) \le \lceil \Delta(G)/2 \rceil + 4$.
(2) *If* $\text{mad}(G) < 3$ *and* $\Delta(G) \ge 9$, *then* $\text{lc}_\ell(G) \le \lceil \Delta(G)/2 \rceil + 2$.
(3) *If* $\text{mad}(G) < 12/5$ *and* $\Delta(G) \ge 3$, *then* $\text{lc}_\ell(G) = \lceil \Delta(G)/2 \rceil + 1$.

Raspaud and Wang [6] conjectured that the bound in Theorem 1(2) holds for all planar graphs with girth at least 6. Since every such graph G has $\text{mad}(G) < 3$, our result proves their conjecture for graphs with $\Delta(G) \ge 9$. Since $\text{mad}(K_{3,3}) = 3$ and $\text{lc}(K_{3,3}) = 5$, we can construct an infinite family of sparse graphs G containing $K_{3,3}$ such that $\text{mad}(G) = 3$, $\Delta(G) = 4$, and $\text{lc}(G) > \lceil \Delta(G)/2 \rceil + 2$. Thus, the maximum degree condition in Theorem 1(2) cannot be lower than 5.

We also note that $\text{lc}(K_{2,3}) = 4 > \lceil \Delta(K_{2,3})/2 \rceil + 1$ and $\text{mad}(K_{2,3}) = 12/5$. Thus, we can construct an infinite family of sparse graphs containing $K_{2,3}$ with maximum degree at most 4. All such graphs have $\text{lc}(G) = \lceil \Delta(G)/2 \rceil + 2$ and can be made sparse enough so that $\text{mad}(G) = \text{mad}(K_{2,3}) = 12/5$. So the bound on $\text{mad}(G)$ in Theorem 1(3) is sharp.

The proofs of our three results all follow the same outline. First, we prove a structural lemma; this says that each graph under consideration must contain at least one from a list of "configurations". Second, we prove that any minimal counterexample to our theorem cannot contain any of these configurations. In this second step, we begin with a linear list coloring of part of the graph, and show how to extend it to the whole graph. As we extend the coloring, we often say that we "choose $c(v) \in L(v)$"; by this we mean that we pick some color $c(v)$ from $L(v)$ and use $c(v)$ to color vertex v. In the following three sections, we will prove our three main results, respectively.

For convenience, we introduce the following notation. A *k-vertex* is a vertex of degree k. A k^+-vertex (k^--vertex) is a vertex of degree at least (at most) k. A k-thread is a path of $k+2$ vertices, where each of the k internal vertices has degree 2, and each of the end vertices has degree at least 3.

2. Planar with girth at least 5 implies $\text{lc}_\ell(G) \le \lceil \frac{\Delta(G)}{2} \rceil + 4$

Lemma 1. *If G is a planar graph with* $\delta(G) \ge 2$ *and with girth at least* 5, *then G contains one of the following two configurations:*

(RC1) *a* 2*-vertex adjacent to a* 5^-*-vertex,*
(RC2) *a* 5*-face with four incident* 3*-vertices and the fifth incident vertex of degree at most* 5.

Proof. We use the discharging method, with initial charge $\mu(f) = d(f) - 5$ for each face f and initial charge $\mu(v) = \frac{3}{2}d(v) - 5$ for each vertex v. By Euler's formula, we have $\sum_{v\in V(G)} \mu(v) + \sum_{f\in F(G)} \mu(f) = (3|E| - 5|V|) + (2|E| - 5|F|) = -5(|F| - |E| + |V|) = -10$. We redistribute charge via the following two discharging rules:

(R1) Each 4^+-vertex v sends charge $\frac{\frac{3}{2}d(v)-5}{d(v)}$ to each incident face (for each time they touch).
(R2) Each face sends charge 1 to each incident 2-vertex and charge $\frac{1}{6}$ to each incident 3-vertex (for each time they touch).

When we write ''for each time they touch'', we mean that if a vertex v is a cut-vertex and appears k times on a facial walk around face f, then the rule for sending charge between v and f should be applied k times. Otherwise, (for example) a 2-vertex that is a cut-vertex would not receive sufficient charge from its one incident face.

Now we will show that if G contains neither configuration (RC1) nor (RC2), then each vertex and each face finishes with nonnegative charge. This is a contradiction, since the discharging rules preserve the sum of the charges (which begins negative). We write $\mu^*(v)$ and $\mu^*(f)$ to denote the charge at vertex v or face f after we apply all discharging rules. If $d(v) = 2$, then $\mu^*(v) = (\frac{3}{2}(2) - 5) + 2(1) = 0$. If $d(v) = 3$, then $\mu^*(v) = (\frac{3}{2}(3) - 5) + 3(\frac{1}{6}) = 0$. By design, each 4^+-vertex finishes with charge 0. So, we now consider the final charge on each face.

Let f be a face of G. For each pair, u_1 and u_2, of adjacent vertices on f, we compute the net charge given from f to u_1 and u_2. If neither of u_1 and u_2 is a 2-vertex, then each vertex receives charge at most $\frac{1}{6}$ from f, so the net charge given from f to u_1 and u_2 is at most $2(\frac{1}{6}) = \frac{1}{3}$. If one of u_1 and u_2, say u_1, is a 2-vertex, then, since G does not contain (RC1), we have $d(u_2) \geq 6$. Hence, the net charge given from f to u_1 and u_2 is at most $1 - \frac{2}{3} = \frac{1}{3}$. (This is true because as the degree of a vertex increases beyond 6, the charge it gives to each incident face increases beyond $\frac{2}{3}$.) By a simple counting argument, we see that the net total charge given from f to all incident vertices is at most $\frac{1}{2}(\frac{1}{3}d(f)) = \frac{1}{6}d(f)$. Since $\mu(f) = d(f) - 5$, we see that $\mu^*(f) \geq 0$ when $d(f) \geq 6$. Now we consider 5-faces.

Suppose f is a 5-face. Let n_2, n_3, and n_{6^+} denote the number of 2-vertices, 3-vertices, and 6^+-vertices incident to f. Note that $\mu^*(f) \geq -n_2 - \frac{1}{6}n_3 + \frac{2}{3}n_{6^+}$. From (RC1), we have $n_2 \leq \lfloor d(f)/2 \rfloor = 2$. If $n_2 = 2$, then $n_3 = 0$ and $n_{6^+} = 3$, so $\mu^*(v) \geq -2 - \frac{1}{6}(0) + \frac{2}{3}(3) = 0$. If $n_2 = 1$, then $n_{6^+} \geq 2$, so $n_3 \leq 2$. Hence, $\mu^*(f) \geq -1 - \frac{1}{6}(2) + \frac{2}{3}(2) = 0$.

Suppose now that f is a 5-face and $n_2 = 0$. Since we have no copy of (RC2), we have either $n_3 = 4$ and $n_{6^+} = 1$, or we have $n_3 \leq 3$. In the first case, we get $\mu^*(f) \geq -0 - \frac{1}{6}(4) + \frac{2}{3}(1) = 0$. In the second case, note that f has at least two 4^+-vertices, each of which gives f charge at least $\frac{1}{4}$. Thus $\mu^*(f) \geq -0 - \frac{1}{6}(3) + \frac{1}{4}(2) = 0$. Hence, every face and every vertex has nonnegative charge. This contradiction completes the proof. □

In Sections 3 and 4, we will only assume bounded maximum average degree (rather than planarity and a girth bound). However, in the proof of the preceding lemma, we needed the stronger hypothesis of planar with girth at least 5. Specifically, we used this hypothesis when considering 5-faces. Our proof relied heavily on the fact that for a 5-face f we have $n_2 \leq \lfloor d(f)/2 \rfloor < d(f)/2$.

Now we use Lemma 1 to prove the following linear list coloring result, which immediately implies Theorem 1(1). For technical reasons, we phrase all of our theorems in terms of an integer M such that $\Delta(G) \leq M$. (Without this technical strengthening, when we consider a subgraph H such that $\Delta(H) < \Delta(G)$, we get complications.) Of course, the interesting case is when $M = \Delta(G)$.

Theorem 2. *Let M be an integer. If G is a planar graph with $\Delta(G) \leq M$ and girth at least 5, then* $\mathrm{lc}_\ell(G) \leq \lceil \frac{M}{2} \rceil + 4$.

Proof. Suppose the theorem is false. Let G be a minimal counterexample and let the list assignment L of size $\lceil \frac{M}{2} \rceil + 4$ be such that G has no linear list coloring from L. Note that G must be connected. Suppose G has a 1-vertex u with neighbor v. By minimality, $G - u$ has a linear list coloring from L. Let $L'(u)$ denote the list of colors in $L(u)$ that neither appear on v, nor appear twice in $N(v)$. Note that $|L'(u)| \geq (\lceil \frac{M}{2} \rceil + 4) - (\lfloor \frac{M-1}{2} \rfloor + 1) = 4$. Thus, if G has a 1-vertex u, we can extend a linear list coloring of $G - u$ to G. So we may assume that $\delta(G) \geq 2$. Since G is a planar graph with $\delta(G) \geq 2$ and girth at least 5, G contains one of the two configurations specified in Lemma 1.

Case (RC1): First, suppose that G contains a 2-vertex u adjacent to a 5^--vertex v. Let w be the other neighbor of u. By minimality, $G - u$ has a linear list coloring from L. In order to avoid creating any 2-colored cycles and to also avoid creating any vertices that have three neighbors with the same color, it is sufficient to avoid coloring u with any color that appears two or more times in $N(v) \cup N(w)$. Furthermore, u must not receive a color used on v or on w. Let $L'(u)$ denote the list of colors in $L(u)$ that may still be used on u. We have $|L'(u)| \geq (\lceil \frac{M}{2} \rceil + 4) - (\lfloor \frac{(M-1)+(5-1)}{2} \rfloor + 2) = (\lceil \frac{M}{2} \rceil + 4) - (\lceil \frac{M}{2} \rceil + 3) = 1$. Thus, we can extend a linear list coloring of $G - u$ to a linear list coloring of G.

Case (RC2): Suppose instead that G contains a 5-face f with four incident 3-vertices and with the fifth incident vertex of degree at most 5. We label the vertices as follows: let u_1, u_2, u_3, and u_4 denote successive 3-vertices, and let v_2 and v_3 denote the neighbors of u_2 and u_3 not on f.

By minimality, $G - \{u_2, u_3\}$ has a linear list coloring from L. Now we will extend the coloring to u_2 and u_3. Let $L'(u_2)$ and $L'(u_3)$ denote the colors in $L(u_2)$ and $L(u_3)$ that are still available for use on u_2 and u_3. When we color u_2, we clearly must avoid the colors on u_1 and v_2. We also want to avoid creating a 2-colored cycle or a vertex that has three neighbors with the same color. To do this, it suffices to avoid any color that appears on two or more vertices at distance two from u_2. This gives us an upper bound on the number of forbidden colors: $2 + \lfloor \frac{(M-1)+2+2}{2} \rfloor = \lceil \frac{M}{2} \rceil + 3$. So $|L'(u_2)| = \lceil \frac{M}{2} \rceil + 4 - (\lceil \frac{M}{2} \rceil + 3) \geq 1$. An analogous count shows that $|L'(u_3)| \geq 1$. However, we might have $L'(u_2) = L'(u_3)$. Thus, we now refine this argument to show that $|L'(u_2)| \geq 2$ or $|L'(u_3)| \geq 2$.

First suppose that $c(u_1) = c(v_2)$. Since the colors on u_1 and v_2 are the same, these two vertices only forbid a single color from use on u_2, rather than the two colors we accounted for above. Thus we get $|L'(u_2)| \geq 2$. As above, $|L'(u_3)| \geq 1$, so we first color u_3, then color u_2 with a color not on u_3. This gives the desired linear coloring of G. Hence, we conclude that $c(u_1) \neq c(v_2)$.

Since $c(u_1) \neq c(v_2)$, when we color u_3, we need not fear creating three neighbors of u_2 with the same color. Further, we need not worry about giving u_3 the same color as either u_1 or v_2, for the following reason. Any 2-colored cycle that contains u_3 and either u_1 or v_2 must also contain u_2 and either u_4 or v_3. Thus, by requiring that u_2 does not get a color that appears on two or more vertices at distance two, we avoid such a 2-colored cycle. So in fact, u_3 only needs to avoid colors that appear on v_3, on u_4, or on at least two vertices of $N(u_4) \cup N(v_3)$. This observation gives us $|L'(u_3)| \geq \left(\left\lceil \frac{M}{2} \right\rceil + 4\right) - \left(\left\lfloor \frac{(M-1)+2}{2} \right\rfloor + 2\right) = \left(\left\lceil \frac{M}{2} \right\rceil + 4\right) - \left(\left\lceil \frac{M}{2} \right\rceil + 2\right) = 2$. So we can color u_2, then color u_3 with a color not on u_2. This gives the desired linear list coloring, and completes the proof. □

A similar, but more detailed, argument proves that if G is a planar graph with girth at least 5 and $\Delta(G) \geq 15$, then $\mathrm{lc}_\ell(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$. A brief sketch of this proof is as follows. First, we can refine Lemma 1 to show that if $\Delta(G) \geq 15$, then in (RC2) at most two neighbors of u_1, u_2, u_3, and u_4 can have high degree. (The key insight is that our present argument only requires that each 6^+-vertex give charge $\frac{2}{3}$ to each incident face; not charge $(\frac{3}{2}d(v) - 5)/d(v)$. Thus, these high degree vertices have lots of extra charge that they can send to adjacent 3-vertices.) With a more careful analysis, we can show that both the original configuration (RC1) and this strengthened version of (RC2) are reducible even with only $\left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$ colors.

3. $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 9$ imply $\mathrm{lc}_\ell(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$

Lemma 2. *If G is a graph with $\mathrm{mad}(G) < 3$, $\delta(G) \geq 2$, and $\Delta(G) \geq 9$, then G contains one of the following five configurations:*

(RC1) *a 2-vertex u adjacent to vertices v and w such that $\left\lceil \frac{d(v)}{2} \right\rceil + \left\lceil \frac{d(w)}{2} \right\rceil < \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$,*
(RC2) *a 3-vertex u adjacent to a 2-vertex and to two other vertices v and w, such that $d(v) + d(w) \leq 8$,*
(RC3) *a 3-vertex adjacent to two 2-vertices,*
(RC4) *a 4-vertex adjacent to four 2-vertices,*
(RC5) *a 5-vertex u that is adjacent to four 2-vertices, each of which is adjacent to another 8^--vertex; and u is also adjacent to a fifth 3^--vertex.*

In fact, the hypothesis $\Delta(G) \geq 9$ cannot be omitted (though the lower bound can possibly be reduced), as we show after we prove the lemma.

Proof. We use discharging, with initial charge $\mu(v) = d(v) - 3$ for each vertex v. Since $\mathrm{mad}(G) < 3$, the sum of the initial charges is negative. Note that only the 2-vertices have negative charge, so we design our discharging rules to pass charge to the 2-vertices. We redistribute the charge via the following three discharging rules:

(R1) Every 4-vertex gives charge $\frac{1}{3}$ to each adjacent 2-vertex.
(R2) Every 5-vertex gives charge $\frac{3}{7}$ to each adjacent 2-vertex that is also adjacent to another 8^--vertex, and it gives charge $\frac{5}{14}$ to every adjacent 3-vertex and every other adjacent 2-vertex.
(R3) Every 6^+-vertex v gives charge $\frac{d(v)-3}{d(v)}$ to each adjacent 2-vertex and 3-vertex.
(R4) Every 3-vertex gives its charge (that it received from rules (R2) and (R3)) to its adjacent 2-vertex (if it has one).

We will show that if G contains none of the five configurations (RC1)–(RC5), then each vertex finishes with nonnegative charge, which is a contradiction. The following observation is an immediate corollary of the fact that G contains no copy of (RC1). We will use this observation below, to show that every vertex finishes with nonnegative charge.

Observation 1. *Suppose that a 2-vertex u has neighbors v and w.*

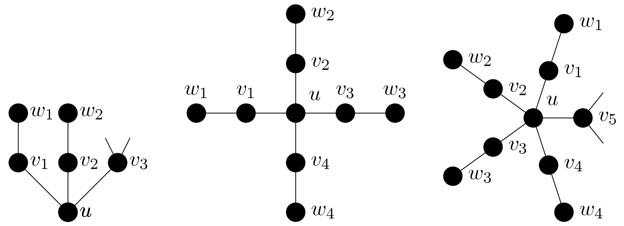
(i) *If $d(v) \in \{3, 4\}$, then $d(w) = \Delta(G)$ if $\Delta(G)$ is odd, and $d(w) \geq \Delta(G) - 1$ if $\Delta(G)$ is even.*
(ii) *If $d(v) \in \{5, 6\}$, then $d(w) \geq \Delta(G) - 2$ if $\Delta(G)$ is odd, and $d(w) \geq \Delta(G) - 3$ if $\Delta(G)$ is even.*

We now use Observation 1 to show that every vertex finishes with nonnegative charge. It is clear from (R3) that every 6^+-vertex finishes with nonnegative charge. The same is true for 3-vertices. So we consider 4-vertices, 5-vertices, and 2-vertices.

Suppose $d(u) = 4$. Since G contains no copy of (RC4), every 4-vertex u is adjacent to at most three 2-vertices. Thus, we have $\mu^*(u) \geq \mu(u) - 3(\frac{1}{3}) = 1 - 3(\frac{1}{3}) = 0$.

Suppose $d(u) = 5$. If u has two or more neighbors that each receive charge at most $\frac{5}{14}$ from u, then $\mu^*(u) \geq \mu(u) - 3(\frac{3}{7}) - 2(\frac{5}{14}) = 2 - \frac{14}{7} = 0$. Similarly, if u has one neighbor that receives no charge from u, then $\mu^*(u) \geq \mu(u) - 4(\frac{3}{7}) > 0$. Hence, we may assume that u sends charge to each neighbor, and that it sends charge $\frac{3}{7}$ to at least four of its neighbors. However, this assumption implies that G contains a copy of configuration (RC5), which is a contradiction.

Finally, suppose $d(u) = 2$. Let the neighbors of u be v and w. Since $\mu(u) = -1$, it suffices to show that u always receives charge at least 1. If $d(v) \geq 6$ and $d(w) \geq 6$, then v and w each give u charge at least $\frac{1}{2}$. So we may assume that $d(v) \leq 5$. Suppose $d(v) = 5$. Since $\Delta(G) \geq 9$, Observation 1 implies that $d(w) \geq 7$. If $d(w) \in \{7, 8\}$, then u receives charge at least $\frac{3}{7} + \frac{4}{7} = 1$. If $d(w) \geq 9$, then u receives charge at least $\frac{5}{14} + \frac{6}{9} > 1$.



(a) (RC3). (b) (RC4). (c) (RC5).

Fig. 1. Configurations (RC3), (RC4), and (RC5) from Lemma 2 and Theorem 3.

If $d(v) = 4$, then Observation 1 implies that $d(w) \geq 9$, so u receives charge at least $\frac{1}{3} + \frac{6}{9} = 1$. If $d(v) = 3$, then the absence of (RC2) implies that at least one neighbor x of v has degree at least 5, so v receives charge at least $\frac{5}{14}$ from x. Since v can have at most one adjacent 2-vertex, u gets charge at least $\frac{5}{14}$ from v. Hence, the total charge that u receives is at least $\frac{6}{9} + \frac{5}{14} > 1$. □

Now we give two examples to show that the hypothesis $\Delta(G) \geq 9$, in Lemma 2 above, cannot be omitted. (We do suspect, however, that this hypothesis can be replaced by $\Delta(G) \geq 7$, or perhaps even by $\Delta(G) \geq 5$.) We first give an example with maximum degree 3. Let G be the dodecahedron, and let E be a matching in G of size 6, such that every face of G contains one edge of E. Form $\widehat{G}$ from G by subdividing each edge of the matching. The girth of $\widehat{G}$ is 6, so (by an easy application of Euler's formula), $\text{mad}(\widehat{G}) < 3$. Despite having $\text{mad}(\widehat{G}) < 3$, $\widehat{G}$ does not contain any of the five configurations (RC1)–(RC5) in Lemma 2. Now we give an example with maximum degree 4. Let G be the octahedron, and let E be a perfect matching in G. Form $\widehat{G}$ from G by subdividing every edge of G except the three edges of E. The average degree of $\widehat{G}$ is $(4 \times 6 + 2 \times 9)/(6 + 9) = \frac{14}{5}$; it is an easy exercise to verify that $\text{mad}(\widehat{G}) = \frac{14}{5}$. Again $\widehat{G}$ contains none of the configurations (RC1)–(RC5).

Now we use Lemma 2 to prove the following linear list coloring result, which immediately implies Theorem 1(2).

Theorem 3. *Let $M \geq 9$ be an integer. If G is a graph with* $\text{mad}(G) < 3$ *and* $\Delta(G) \leq M$, *then* $\text{lc}_\ell(G) \leq \lceil \frac{M}{2} \rceil + 2$.

Proof. Suppose the theorem is false. Let G be a minimal counterexample and let the list assignment L of size $\lceil \frac{M}{2} \rceil + 2$ be such that G has no linear list coloring from L. Since $M \geq 9$, we have $|L(v)| = \lceil \frac{M}{2} \rceil + 2 \geq 7$ for every $v \in V$. Note that G must be connected. Suppose G has a 1-vertex u with neighbor v. By minimality, $G - u$ has a linear list coloring from L. Let $L'(u)$ denote the list of colors in $L(u)$ that neither appear on v, nor appear twice in $N(v)$. Note that $|L'(u)| \geq \left(\lceil \frac{M}{2} \rceil + 2\right) - \left(\lfloor \frac{M-1}{2} \rfloor + 1\right) = 2$. Thus, if G has a 1-vertex u, we can extend a linear list coloring of $G - u$ to G. So we may assume that $\delta(G) \geq 2$.

Since G is a graph with $\delta(G) \geq 2$ and $\text{mad}(G) < 3$, G contains one of the five configurations (RC1)–(RC5) specified in Lemma 2. We consider each of these five configurations in turn, and in each case we construct a linear coloring of G from L.

Case (RC1): Suppose that G contains configuration (RC1). Let u be a 2-vertex adjacent to vertices v and w such that $\lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil < \lceil \frac{M}{2} \rceil + 2$. By the minimality of G, subgraph $G - u$ has a linear list coloring c.

If $c(v) \neq c(w)$, then u can receive any color except for $c(v)$, $c(w)$, and those colors that appear twice on $N(v)$ or twice on $N(w)$. So the number of colors forbidden is at most $2 + \lfloor \frac{d(v)-1}{2} \rfloor + \lfloor \frac{d(w)-1}{2} \rfloor = \lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil$. Since $|L(u)| = \lceil \frac{M}{2} \rceil + 2$, and since $\lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil < \lceil \frac{M}{2} \rceil + 2$, we can extend the coloring to u. So we assume instead that $c(v) = c(w) = 1$.

If $c(v) = c(w)$, then (similar to that above), u can receive any color except for $c(v)$ and those colors that appear twice on $N(v) \cup N(w)$. The number of forbidden colors is at most $1 + \lfloor \frac{(d(v)-1)+(d(w)-1)}{2} \rfloor \leq \lceil \frac{d(v)}{2} \rceil + \lceil \frac{d(w)}{2} \rceil$. So, once again, we can extend the coloring to u.

Case (RC2): Suppose that G contains configuration (RC2). Let u be a 3-vertex adjacent to a 2-vertex and to two other neighbors v and w with $d(v) + d(w) \leq 8$. By the minimality of G, subgraph $G - u$ has a linear list coloring from L. If all three neighbors of u have the same color, then we will not get a linear coloring of G no matter how we color u. In this case, we can recolor the 2-vertex and still have a linear coloring of $G - u$. Now we will extend the coloring to u.

Let $L'(u)$ denote the colors in $L(u)$ that are still available for use on u. When we color u, we clearly must avoid the colors on its three neighbors. We also want to avoid creating a 2-colored cycle or a vertex that has three neighbors with the same color. To do this, it suffices to avoid any color that appears on two or more vertices at distance two from u. This gives us an upper bound on the number of forbidden colors: $3 + \lfloor \frac{(d(v)-1)+(d(w)-1)+1}{2} \rfloor = 3 + \lfloor \frac{d(v)+d(w)-1}{2} \rfloor \leq 3 + \lfloor \frac{7}{2} \rfloor = 6$. Since $|L(u)| \geq 7$, we have $|L'(u)| \geq 1$. Thus, we can extend the coloring to u.

Case (RC3): Suppose that G contains configuration (RC3), shown in Fig. 1. Let u be a 3-vertex that has neighbors v_1, v_2, and v_3 with $d(v_1) = d(v_2) = 2$ and $d(v) = 3$. Let $N(v_i) = \{w_i, u\}$ for $i \in \{1, 2\}$. By the minimality of G, subgraph $G - \{u, v_1, v_2\}$ has a linear list coloring c from L. For each uncolored vertex $z \in \{u, v_1, v_2\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on z. Note that $|L'(z)| \geq 2$ for each uncolored vertex z.

Suppose that $L'(u) = \{c(w_1), c(w_2)\}$; this means that $c(v_3) \notin \{c(w_1), c(w_2)\}$. Color u with $c(w_1)$. Now choose $c(v_1) \in L'(v_1) - c(v_3)$ and $c(v_2) \in L'(v_2) - c(w_1)$. This is a valid linear coloring of G.

Suppose instead that $L'(u) \setminus \{c(w_1), c(w_2)\} \neq \emptyset$. Choose $c(u) \in L'(u) - \{c(w_1), c(w_2)\}$, choose $c(v_1) \in L'(v_1) - c(u)$, and choose $c(v_2) \in L'(v_2) - c(u)$. This coloring is proper and contains no 2-alternating path through u. Hence, it is a linear coloring unless $c(v_1) = c(v_2) = c(v_3)$. If no other choice of $c(v_1)$ and $c(v_2)$ can avoid this problem, then we can conclude that $L'(v_1) = L'(v_2) = \{c(v_3), c_1\}$ (for some color c_1); further $L'(u) - \{c(w_1), c(w_2)\} = \{c_1\}$. Suppose we are in this case.

If $c(w_1) \neq c(w_2)$, then, without loss of generality, $L'(u) = \{c(w_1), c_1\}$. Now let $c(u) = c(w_1)$, $c(v_1) = c_1$, and $c(v_2) = c(v_3)$ This is a valid linear coloring. So, by relabeling, we may assume that $c(w_1) = c(w_2) = 1$, $c(v_3) = 2$, and $c_1 = 3$. Thus $L'(v_1) = L'(v_2) = \{2, 3\}$ and $L'(u) = \{1, 3\}$.

Note that $\{2, 3\} \subseteq L'(v_i)$ implies that 2 and 3 each appear at most once in $N(w_i)$ (for $i \in \{1, 2\}$). If 3 does not appear on both $N(w_1)$ and $N(w_2)$, then let $c(v_1) = c(v_2) = 3$ and $c(u) = 1$. If 2 does not appear on both $N(w_1)$ and $N(w_2)$, then let $c(u) = 1$, $c(v_1) = 2$, $c(v_2) = 3$ (or $c(u) = 1$, $c(v_1) = 3$, $c(v_2) = 2$). So, we can assume that 2 and 3 each appear once on both $N(w_1)$ and $N(w_2)$. However, now $|L'(v_i)| \geq (\lceil \frac{M}{2} \rceil + 2) - (\lfloor \frac{M-3}{2} \rfloor + 1) \geq 3$, which is a contradiction.

Case (RC4): Suppose that G contains configuration (RC4), shown in Fig. 1. Let u be a 4-vertex and let $N(u) = \{v_i : 1 \leq i \leq 4 \text{ such that } d(v_i) = 2\}$. Also let $N(v_i) = \{u, w_i\}$ for $1 \leq i \leq 4$. By the minimality of G, subgraph $G - \{u, v_1, v_2, v_3, v_4\}$ has a linear list coloring from L. For each uncolored vertex z, let $L'(z)$ denote the list of colors still available for z. Note that $|L'(v_i)| \geq 2$ and $|L'(u)| = |L(u)| = \lceil \frac{M}{2} \rceil + 2 \geq 7$, since $M \geq 9$.

We can color the v_i's from their lists so that every color is used on at most two v_i's, as follows. If some color c is available for use on two or more v_i's, then use c on exactly two of them, and color each of the remaining v_i's with another color (which could be the same for both of them). Otherwise, all the v_i's have disjoint lists of available colors, so color them arbitrarily.

If the four colors on the v_i's are all distinct, then color u with a fifth color. If $c(v_1) = c(v_2)$ but $c(v_1)$, $c(v_3)$, and $c(v_4)$ are all distinct, then choose $c(u)$ so that $c(u) \notin \{c(v_1), c(v_3), c(v_4), c(w_1)\}$. Finally, if $c(v_1) = c(v_2)$ and $c(v_3) = c(v_4)$ (which together imply $c(v_1) \neq c(v_3)$), then choose $c(u)$ so that $c(u) \notin \{c(v_1), c(v_3), c(w_1), c(w_3)\}$.

Case (RC5): Suppose that G contains configuration (RC5), shown in Fig. 1. Let u be a 5-vertex and let $N(u) = \{v_i : 1 \leq i \leq 5\}$, such that $d(v_i) = 2$ for $1 \leq i \leq 4$ and $d(v_5) \leq 3$. Also let $N(v_i) = \{u, w_i\}$ for $1 \leq i \leq 4$, where $d(w_i) \leq 8$. By the minimality of G, subgraph $G - \{u, v_1, v_2, v_3, v_4\}$ has a linear coloring c from L. For each uncolored vertex $z \in \{u, v_1, v_2, v_3, v_4\}$, let $L'(z)$ denote the list of colors still available for z. Since $d(w_i) \leq 8$, we have $|L'(v_i)| \geq 3$. Conversely, $|L'(u)| \geq \lceil \frac{M}{2} \rceil + 2 - (\lfloor \frac{2}{2} \rfloor + 1) = \lceil \frac{M}{2} \rceil \geq 5$, since $M \geq 9$. Now let $L''(v_i) = L'(v_i) - c(v_5)$; note that $|L''(v_i)| \geq 2$. We now extend the coloring by using the lists $L'(u)$ and $L''(v_i)$. We can completely ignore v_5 (since we deleted $c(v_5)$ from the lists), so the analysis is exactly the same as in Case (RC4). □

As we explained in the Introduction, this theorem immediately yields the following corollary.

Corollary 1. *If graph G is planar, has girth at least 6, and $\Delta(G) \geq 9$, then* $\text{lc}_\ell(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 2$.

Although our proof of Theorem 3 relies heavily on the hypothesis $\Delta(G) \geq 9$, we suspect that the Theorem is true even when this hypothesis is removed. Namely, we conjecture that every graph G with $\text{mad}(G) < 3$ satisfies $\text{lc}_\ell(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 2$. If true, this result is best possible, as shown by the graph $K_{3,3}$, since $\text{lc}_\ell(K_{3,3}) = 5$. Furthermore, every graph G with $K_{3,3} \subseteq G$, $\text{mad}(G) = 3$, and $\Delta(G) \in \{3, 4\}$ shows that this result is best possible.

4. $\text{mad}(G) < \frac{12}{5}$ implies $\text{lc}_\ell(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$

In this section, we prove that if G is a graph with $\Delta(G) \geq 3$ and $\text{mad}(G) < \frac{12}{5}$, then $\text{lc}_\ell(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$. For such graphs, we prove an upper bound that matches the trivial lower bound $\text{lc}_\ell(G) \geq \lceil \frac{\Delta(G)}{2} \rceil + 1$. Recall (from the Introduction) that our bound on $\text{mad}(G)$ is best possible, as demonstrated by $K_{2,3}$, since $\text{mad}(K_{2,3}) = \frac{12}{5}$ and $\text{lc}_\ell(K_{2,3}) > \lceil \frac{\Delta(K_{2,3})}{2} \rceil + 1$.

Lemma 3. *If G is a graph with* $\text{mad}(G) < \frac{12}{5}$ *and* $\delta(G) \geq 2$, *then G contains one of the following four configurations:*

(RC1) *a 3^+-thread,*
(RC2) *a 3-vertex v incident to two 1^+-threads and one 2-thread, such that the vertex at distance two from v along each 1^+-thread is a 3^--vertex,*
(RC3) *adjacent 3-vertices with at least seven 2-vertices in their incident threads,*
(RC4) *a path of three vertices uwv with $d(u) = d(w) = d(v) = 3$ such that w is incident to a 2-thread and u and v are each incident to two 2-threads.*

Proof. We use discharging, with initial charge $\mu(v) = d(v) - \frac{12}{5}$ for each vertex v. Since $\text{mad}(G) < \frac{12}{5}$, the sum of the initial charges is negative. We use the following three discharging rules:

(R1) Every 2-vertex gets charge $\frac{1}{5}$ from each of the endpoints of its thread.
(R2) Every 3-vertex incident to two 2-threads gets charge $\frac{1}{5}$ from its 3^+-neighbor.
(R3) Every 3-vertex incident to a 1-thread gets charge $\frac{1}{5}$ from the other endpoint of the 1-thread if it is a 4^+-vertex.

Now we will show that if G contains none of the configurations (RC1)–(RC4), then every vertex finishes with nonnegative charge, which is a contradiction. If $d(v) = 2$, then $\mu^*(v) = d(v) - \frac{12}{5} + 2(\frac{1}{5}) = 0$. If $d(v) \geq 4$, then, since G contains no

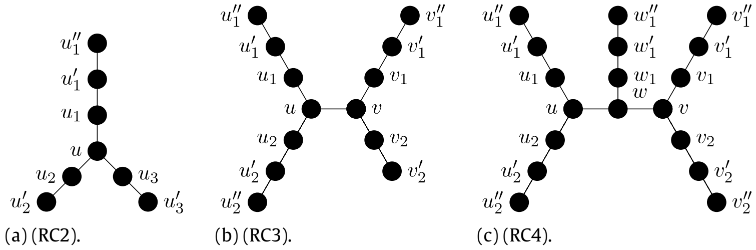


Fig. 2. Configurations (RC2), (RC3), and (RC4) from Lemma 3 and Theorem 4.

3^+-threads (by (RC1)), v gives away charge $\frac{1}{5}$ to each of at most $2d(v)$ 2-vertices. Note further that if v gives away charge $\frac{1}{5}$ to t 3-vertices via (R2) and/or (R3), for some constant t, then v gives away charge $\frac{1}{5}$ to at most $2d(v) - t$ 2-vertices. Thus, we have $\mu^*(v) \geq d(v) - \frac{12}{5} - \frac{1}{5}(2d(v)) = \frac{3}{5}(d(v) - 4) \geq 0$. So we only need to consider 3-vertices.

Let $d(v) = 3$. Suppose v has at most three 2-vertices in its incident threads. If v does not give away charge by (R2), then v gives away charge at most $3(\frac{1}{5})$, so $\mu^*(v) \geq 3 - \frac{12}{5} - 3(\frac{1}{5}) = 0$. If v does give charge by (R2), then, since G contains no copy of (RC3), v has at most two 2-vertices in its incident threads. Thus v gives away charge at most $3(\frac{1}{5})$, unless both v is incident to a 2-thread and also v gives away charge by (R2) to two distinct vertices. However, this situation cannot occur, since it implies that G contains a copy of (RC4), which is a contradiction.

Suppose instead that v has at least four 2-vertices in its incident threads. Since G contains no copy of (RC2), either v is incident to two 2-threads and also adjacent to a 3^+-vertex, or v is incident to two 1-threads and one 2-thread and the other end of at least one 1-thread is a 4^+-vertex. In each case, v gives away charge $4(\frac{1}{5})$ and receives charge at least $\frac{1}{5}$, so $\mu^*(v) \geq 3 - \frac{12}{5} - 4(\frac{1}{5}) + \frac{1}{5} = 0$. $\square$

Now we use Lemma 3 to prove the following linear list coloring result.

Theorem 4. *Let $M \geq 3$ be an integer. If G is a graph with $\mathrm{mad}(G) < \frac{12}{5}$ and $\Delta(G) \leq M$, then $\mathrm{lc}_\ell(G) = \lceil \frac{M}{2} \rceil + 1$.*

Proof. Suppose the theorem is false. Let G be a minimal counterexample and let list assignment L, of size $\lceil \frac{M}{2} \rceil + 1$, be such that G has no linear list coloring from L. Since $M \geq 3$, we have $|L(v)| = \lceil \frac{M}{2} \rceil + 1 \geq 3$ for all $v \in V$. Note that G must be connected. Suppose that G contains a 1-vertex u with neighbor v. By the minimality of G, subgraph $G - \{u\}$ has a linear list coloring from L. Let $L'(u)$ denote the list of colors in $L(u)$ that neither appear on v nor appear twice in $N(v)$. Note that $|L'(u)| \geq (\lceil \frac{M}{2} \rceil + 1) - \lfloor \frac{M-1}{2} \rfloor - 1 \geq 1$. Thus, if G has a 1-vertex u, we can extend a linear list coloring of $G - u$ to G. So we may assume that $\delta(G) \geq 2$.

Since G has $\delta(G) \geq 2$ and $\mathrm{mad}(G) < \frac{12}{5}$, G contains one of the four configurations specified in Lemma 3. We consider each of these four configurations in turn, and in each case we construct a linear coloring of G from L.

Case (RC1): Suppose that G contains (RC1): a 3^+-thread. Let u, u_1, u_2, u_3, u_4 be part of the thread, that is, $d(u) \geq 3$, $d(u_1) = d(u_2) = d(u_3) = 2$, and $d(u_4) \geq 2$. By the minimality of G, subgraph $G - \{u_2\}$ has a linear coloring from L. If $c(u_1) = c(u_3)$, then $|L(u_2)| \geq 2$, so we choose $c(u_2) \in L(u_2) - \{c(u)\}$. If $c(u_1) \neq c(u_3)$, then $|L(u_2)| \geq 1$, so we choose $c(u_2) \in L(u_2)$. Note that either $c(u_2) \neq c(u)$ or $c(u_1) \neq c(u_3)$, so we have not created a 2-colored cycle.

Case (RC2): Suppose instead that G contains (RC2), shown in Fig. 2. Let u be a 3-vertex that is incident to one 2-thread u, u_1, u_1', u_1'' with $d(u_1'') \geq 3$ and incident to two 1^+-threads u, u_2, u_2' and u, u_3, u_3' with $2 \leq d(u_2') \leq 3$ and $2 \leq d(u_3') \leq 3$. By the minimality of G, subgraph $G - \{u, u_1, u_2, u_3\}$ has a linear coloring from L. Now we will extend the coloring to G.

For each uncolored vertex $z \in \{u, u_1, u_2, u_3\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on z. When we extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. Note that $|L'(u_1)| \geq 2$, $|L'(u_2)| \geq 1$, and $|L'(u_3)| \geq 1$.

Suppose $|L'(u_2) \cup L'(u_3)| \geq 2$. We choose $c(u_2) \in L'(u_2)$ and $c(u_3) \in L'(u_3)$ such that $c(u_2) \neq c(u_3)$. Next we choose $c(u) \in L'(u) - \{c(u_2), c(u_3)\}$. If $c(u) \neq c(u_1')$, then we choose $c(u_1) \in L'(u_1) - \{c(u)\}$. If instead $c(u) = c(u_1')$, then we choose $c(u_1) \in L'(u_1) - \{c(u_1'')\}$. This gives a valid linear coloring.

Suppose instead that $|L'(u_2) \cup L'(u_3)| = 1$. Thus $L'(u_2) = L'(u_3) = \{a\}$, for some color a. Clearly, we must choose $c(u_2) = c(u_3) = a$. Note that this happens only if both $d(u_2') = d(u_3') = 3$ and the two other neighbors of u_2' (and u_3') have the same color. Now we choose $c(u_1) \in L(u_1) - \{a, c(u_1')\}$ and $c(u) \in L(u) - \{a\}$.

Since $c(u_1) \neq a$, we have not created any vertex with 3 neighbors of the same color, and we have not created any 2-colored cycle passing through u_1. Since $c(u_2)$ does not appear on the other neighbors of u_2', we have not created any 2-colored cycle passing through u_2.

Case (RC3): Now suppose instead that G contains (RC3): two adjacent 3-vertices with at least seven 2-vertices in their incident threads (shown in Fig. 2). We label the vertices as follows: let u and v be the adjacent 3-vertices, u is incident to two 2-threads u, u_1, u_1', u_1'' and u, u_2, u_2', u_2'' and v is incident to one 2-thread v, v_1, v_1', v_1'' and one 1^+-thread v, v_2, v_2'.

By the minimality of G, subgraph $G - \{u, v, u_1, u_2, v_1\}$ has a linear coloring from L. Now we will extend the coloring to G. For each vertex $z \in \{u, v, u_1, u_2, v_1\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on z. When we extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. Note that $|L'(u_1)| \geq 2$, $|L'(u_2)| \geq 2$, $|L'(v_1)| \geq 2$, $|L'(u)| \geq 3$, and $|L'(v)| \geq 3$; we may assume that equality holds in each case.

Since $|L'(u)| = 3 > 2 = |L'(u_1)|$, we can choose $c(u) \in L'(u) - L'(u_1)$. If $c(u) = c(v_2)$, then choose $c(v_1) \in L'(v_1) - \{c(u)\}$ and $c(v) \in L'(v) - \{c(v_1)\}$. If instead $c(u) \neq c(v_2)$, then choose $c(v) \in L'(v) - \{c(u)\}$.

Now if $c(v) \neq c(v_1')$, then choose $c(v_1') \in L'(v_1) - \{c(v)\}$; if $c(v) = c(v_1')$, then choose $c(v_1') \in L'(v_1) - \{c(v_1'')\}$. Next, choose $c(u_1) \in L'(u_1) - \{c(v)\}$. Finally, if $c(u) = c(u_2')$, then choose $c(u_2) \in L'(u_2) - \{c(u_2'')\}$; otherwise, choose $c(u_2) \in L'(u_2) - \{c(u)\}$.

Recall that $c(u_1) \neq c(v)$ and either $c(u) \neq c(v_2)$ or $c(v_1) \neq c(v_2)$; thus, we do not create any vertices with three neighbors of the same color. By construction, we have no 2-colored cycles through u_2 or v_1. Further, $c(u_1) \neq c(v)$, so we do not create any 2-colored cycles.

Case (RC4): Suppose that G contains (RC4). We label the vertices as follows: let u, w, v be the path; let u, u_1, u_1', u_1'' and u, u_2, u_2', u_2'' be the 2-threads incident to u; let v, v_1, v_1', v_1'' and v, v_2, v_2', v_2'' be the 2-threads incident to v; and let w, w_1, w_1', w_1'' be the 2-thread incident to w.

By the minimality of G, subgraph $G - \{u, u_1, u_2, v, v_1, v_2, w, w_1\}$ has a linear coloring from L. Now we will extend the coloring to G. For each vertex $z \in \{u, u_1, u_2, v, v_1, v_2, w, w_1\}$, let $L'(z)$ denote the colors in $L(z)$ that are still available for use on z. When we extend the coloring, we obviously must get a proper coloring. In addition, we must avoid creating 2-colored cycles and avoid creating vertices with 3 neighbors of the same color. We will show explicitly how to color u, u_1, u_2, w, and w_1 (and we will color v, v_1, and v_2, analogously). We consider two subcases. In fact, we may have one "side" (u, u_1, u_1', u_2, and u_2') that is in Subcase (i) and the other side that is in Subcase (ii); this is not a problem, since we color the sides independently.

Subcase (i): Suppose that $c(u_1') = c(u_2')$. If $c(u_1') \notin L'(u)$, then we can choose $c(u_1) \in L'(u_1)$ and $c(u_2) \in L'(u_2)$ such that $c(u_1) \neq c(u_2)$, and afterward we choose $c(u) \in L'(u) - \{c(u_1'), c(u_1), c(u_2)\}$. If $c(u_1') \in L'(u)$, then let $c(u) = c(u_1')$. Choose $c(v)$ analogously. In this instance, we wait to choose $c(u_1)$ and $c(u_2)$ until after we choose $c(w)$.

If $c(u) = c(v)$, then choose $c(w_1) \in L'(w_1) - \{c(u)\}$ and $c(w) \in L'(w) - \{c(w_1), c(u)\}$. If $c(u) \neq c(v)$, then choose $c(w) \in L'(w) - \{c(u), c(v)\}$ and $c(w_1) \in L'(w_1) - \{c(w)\}$. Finally, choose $c(u_1) \in L'(u_1) - \{c(u), c(u_1'')\}$ and $c(u_2) \in L'(u_2) - \{c(u), c(w)\}$ (if we have not chosen these colors yet; recall that $c(u) = c(u_1')$, so $c(u_1) \neq c(u)$; analogously, $c(u_2) \neq c(w)$).

Subcase (ii): $c(u_1') \neq c(u_2')$. Choose $c(u) \in L'(u) - \{c(u_1'), c(u_2')\}$. Choose $c(v)$ analogously. Now color w and w_1 as above. Finally, we will color u_1, u_2, v_1, and v_2, as below.

If we can, we choose $c(u_1) \in L'(u_1) - \{c(u)\}$, and $c(u_2) \in L'(u_2) - \{c(u)\}$ such that either $c(u_1) \neq c(w)$ or $c(u_2) \neq c(w)$. If this is impossible, then $L'(u_1) = L'(u_2) = \{c(u), c(w)\}$; furthermore, $L(u) = \{c(u), c(u_1'), c(u_2')\}$. Now let $c(u_1) = c(u_2) = c(u)$ and *recolor* u with a new color in $L'(u) - \{c(u_1), c(w_1), c(w)\}$ (note that $c(w) \notin L'(u)$). Finally, color v_1, v_2, and v analogously.

It is clear that we have created a proper coloring. It is also straightforward to verify that we did not create any vertices with 3 neighbors of the same color, and we did not create any 2-colored cycles. □

This theorem immediately yields the following corollary.

Corollary 2. *If graph G is planar with girth at least* 12 *and* $\Delta(G) \geq 3$, *then* $\mathrm{lc}_\ell(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

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